Charged analogue of Finch-Skea stars

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Abstract

We present solutions to the Einstein-Maxwell system of equations in spherically symmetric gravitational fields for static interior spacetimes with a specified form of the electric field intensity. The condition of pressure isotropy yields three category of solutions. The first category is expressible in terms of elementary functions and does not have an uncharged limit. The second category is given in terms of Bessel functions of half-integer order. These charged solutions satisfy a barotropic equation of state and contain Finch-Skea uncharged stars. The third category is obtained in terms of modified Bessel functions of half-integer order and does not have an uncharged limit. The physical features of the charged analogue of the Finch-Skea stars are studied in detail. In particular the condition of causality is satisfied and the speed of sound does not exceed the speed of light. The physical analysis indicates that this analogue is a realistic model for static charged relativistic perfect fluid spheres.

1 Introduction

In the past many classes of exact solutions of the Einstein field equations have been found for static spherically symmetric gravitational fields with isotropic matter. A comprehensive list of Einstein solutions is provided by Delgaty and Lake [1]. These solutions may be used to model a neutral relativistic star as they are matchable to the Schwarzschild exterior at the boundary. In comparison fewer exact solutions of the Einstein-Maxwell field equations are known for static spherically symmetric gravitational fields with isotropic matter with nonzero electric fields. A recent review of Einstein-Maxwell solutions is given by Ivanov [2]. These solutions may be utilized to model a charged relativistic star as they match to the Reissner-Nordstrom exterior.

In modelling a charged relativistic sphere it is desirable to ensure that two general features are contained in the model. Firstly the model should be physically reasonable: the gravitational, electromagnetic and matter variables are continuous and well behaved in the interior, the interior metric matches smoothly to the exterior spacetime and causality is not violated. Secondly we should regain an uncharged solution (which should also satisfy the relevant physical conditions) of the the Einstein equations when the electrical field vanishes; a neutral star should be regainable as a stable equilibrium end state. The anisotropic charged model of Sharma et al [3] is an example that satisfies the stated criteria. An approach in satisfying the two criteria is to construct the model such that the limiting uncharged solution is a known exact solution. This is not easy to achieve in practice as the number of known exact uncharged solutions satisfying all conditions of physical acceptability are limited as established by Delgaty and Lake [1].

Our objective in the paper is to generate a new solution of the Einstein-Maxwell system that is physically acceptable and necessarily contains a neutral stellar model. The neutral stellar model is the Finch and Skea star [4] which satisfies all the requirement of physical acceptability. In §2 we express the Einstein equations for neutral matter and the Einstein-Maxwell system for charged matter as equivalent sets of differential equations utilising a transformation due to Durgapal and Bannerji [5]. We choose particular forms for one of the gravitational potentials and the the electric field intensity in §3. The condition of pressure isotropy becomes a second order linear equation in the remaining gravitational potential. We integrate the condition of pressure isotropy and consequently produce three classes of exact solutions to the Einstein-Maxwell field equations which can be written explicitly in terms of elementary functions as shown in §3.1, §3.2 and §3.3. We regain the uncharged exact solution found previously from the charged analogue of the Finch and Skea model in §3.2; we also demonstrate that the charged solution satisfies a barotropic equation of state. In §4 we comprehensively study the physical features of the charged Finch-Skea model. Graphs are generated for particular parameter values for the gravitational, electromagnetic and matter variables. We consider in particular the gravitational behaviour at the centre, and also produce a specific value for the the charge to radius ratio. We believe that this detailed analysis of the matter variables represents a realistic model of a compact charged object. Some brief concluding remarks are made in §5.

2 Einstein-Maxwell equations

We assume that the interior of a spherically symmetric relativistic star is described by the line element

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

in Schwarzschild coordinates (t, r, θ, ϕ) where the functions $\nu(r)$ and $\lambda(r)$ are gravitational potentials. For neutral perfect fluids the Einstein field equations can be expressed as follows

$$\begin{aligned}
\frac{1}{r^2}[r(1-e^{-2\lambda})]' &= \rho && \text{(2a)}\\
-\frac{1}{r^2}\,(1-e^{-2\lambda}) + \frac{2\nu'}{r}\,e^{-2\lambda} &= p && \text{(2b)}\\
e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) &= p && \text{(2c)}
\end{aligned}$$

for the geometry described by (1). We measure the energy density ρ and pressure p relative to the comoving fluid 4-velocity $u^a = e^{-\nu}\delta^a_0$ and primes denote differentiation with respect to the radial coordinate r. We utilize units where the coupling constant $\frac{8\pi G}{c^4} = 1$ and the speed of light $c = 1$. An equivalent form of the field equations is obtained if we introduce the transformation

$$A^2 y^2(x) = e^{2\nu(r)}, \quad Z(x) = e^{-2\lambda(r)}, \quad x = Cr^2 \tag{3}$$

where the quantities A and C are arbitrary constants. Under the transformation (3) the system (2) has the form

$$\begin{aligned}
\frac{1-Z}{x} - 2\dot{Z} &= \frac{\rho}{C} && \text{(4a)}\\
4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} &= \frac{p}{C} && \text{(4b)}\\
4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1)y &= 0 && \text{(4c)}
\end{aligned}$$

where the dots denotes differentiation with respect to the variable x.

A generalisation of the system (4) is the coupled Einstein-Maxwell field equations given by

$$\begin{aligned}
\frac{1-Z}{x} - 2\dot{Z} &= \frac{\rho}{C} + \frac{E^2}{2C} && \text{(5a)}\\
4Z\frac{\dot{y}}{y} + \frac{Z-1}{x} &= \frac{p}{C} - \frac{E^2}{2C} && \text{(5b)}\\
4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + \left(\dot{Z}x - Z + 1 - \frac{E^2 x}{C}\right)y &= 0 && \text{(5c)}\\
\frac{\sigma^2}{C} &= \frac{4Z}{x}\left(x\dot{E} + E\right)^2 && \text{(5d)}
\end{aligned}$$

where E is the electric field intensity and σ is the proper charge density. When the electric field $E = 0$ then the Einstein-Maxwell equations (5) reduce to the Einstein equations (4) for neutral matter. The system of equations (5) governs the behaviour of the gravitational field for a charged perfect fluid. The particular representation of the Einstein-Maxwell system as given in (5) may be easier to integrate in certain situations as demonstrated by Thirukkanesh and Maharaj [6]. An interior stellar solution of (5), for the line element (1), should match to the exterior gravitational field described by the Reissner-Nordstrom line element. In terms of Schwarzschild coordinates the Reissner-Nordstrom solution has the form

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{6}$$

where M and Q are associated with the mass and charge of the sphere respectively.

3 Exact Solutions

It is evident that the system of equations (5) is underdetermined. Consequently we need to specify two of the variables in advance so that a solution can be obtained. A possible approach is to specify an equation of state relating ρ to p and additionally to specify one of the gravitational potentials. This leads to differential equations that are highly nonlinear and difficult to integrate. Another option, which is pursued here, is to postulate a form for the gravitational potential Z and to choose a form for the electrostatic field E. Our choice for the function $Z(x)$ is contained in the form used by Thirukkanesh and Maharaj [6] and Maharaj and Mkhwanazi [7] which ensures that we regain as a special case models of fluid spheres with uncharged matter distributions analysed previously.

The metric function $Z(x)$ is chosen to be of the form

$$Z = (1+x)^n \tag{7}$$

where $n \neq 0$ is a real number. (If $n = 0$ then (5a) implies that $\rho = -E^2/2$ which is negative.) The form (7) ensures that the metric function e^λ behaves as

$$e^\lambda = 1 + O(r^2)$$

near $r = 0$ for suitable choices of n. In fact this is a sufficient condition for a static perfect fluid sphere to be regular at the centre as pointed out by Maartens and Maharaj [8]. We choose $n = -1$ in (7) to complete the integration. Additionally we postulate the form

$$\frac{E^2}{C} = \frac{\alpha x}{(1+x)^2} \tag{8}$$

for E which depends on the real valued parameter α. The form (8) is physically palatable since E^2 remains regular and positive throughout the sphere if $\alpha > 0$. In addition the field intensity E becomes zero at the stellar centre and attains a maximum value of $E = \sqrt{\alpha C/2}$ when $r = 1/\sqrt{C}$. In what follows the Einstein-Maxwell equations (5) are solved for all nonnegative values of the parameter α.

With the choices (7) (taking $n = -1$) and (8), equation (5c) reduces to

$$4(1+x)\ddot{y} - 2\dot{y} + (1-\alpha)\, y = 0 \tag{9}$$

It is convenient to categorise our solutions in terms of different values of α. We consider, in turn, the following three cases:

$$\alpha = 1, \;\; 0 \leq \alpha < 1, \;\; \alpha > 1$$

which generate classes of solutions to the Einstein–Maxwell system (5).

3.1 The case $\alpha = 1$

With the choice $\alpha = 1$, equation (9) assumes the simpler form

$$4(1+x)\ddot{y} - 2\dot{y} = 0$$

which is an ordinary differential equation of reducible order. It is easily integrated to yield

$$y = \frac{2}{3}c_1(1+x)^{3/2} + c_2$$

where c_1 and c_2 are constants of integration to be determined from physical considerations. The corresponding expressions for ρ and p can then be established via (5a) and (5b) respectively. The complete solution to the Einstein–Maxwell field equations (5) is then given by

$$e^{2\lambda} = 1 + Cr^2 \tag{10a}$$

$$e^{2\nu} = A^2 \left[\frac{2}{3}c_1(1+Cr^2)^{3/2} + c_2\right]^2 \tag{10b}$$

$$\frac{\rho}{C} = \frac{Cr^2 + 6}{2(1+Cr^2)^2} \tag{10c}$$

$$\frac{p}{C} = \frac{c_1(10 - Cr^2)(1+Cr^2)^{3/2} - c_2(2 + Cr^2)}{2(1+Cr^2)\left[c_1(1+Cr^2)^{3/2} + 3c_2\right]} \tag{10d}$$

$$E^2 = \frac{C^2r^2}{(1+Cr^2)^2} \tag{10e}$$

$$\sigma^2 = \frac{C^2(3+Cr^2)}{(1+Cr^2)^5} \tag{10f}$$

Note that this charged solution does not have an uncharged analogue as the electrostatic field intensity E cannot vanish (except at the centre). This effect essentially results from our condition that $\alpha = 1$. The line element for the solution (10) is given by

$$ds^2 = -A^2\left[\frac{2}{3}c_1(1+Cr^2)^{3/2} + c_2\right]^2 dt^2 + (1+Cr^2)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{11}$$

In spite of the simplicity of this solution to the Einstein–Maxwell equations, it does not seem to have been published before.

3.2 The case $0 \leq \alpha < 1$

With $0 \leq \alpha < 1$ equation (9) is more difficult to solve. However it can be transformed to a standard Bessel equation. It is convenient to introduce the substitution $X = 1 + x$ in (9) to yield

$$4X\frac{d^2Y}{dX^2} - 2\frac{dY}{dX} + (1-\alpha)Y = 0 \tag{12}$$

where $y(X) = Y$. We now introduce a new function $u(X)$ such that $Y(X) = u(X)X^m$ where m is a real number. Then the ordinary differential equation (12) becomes

$$4X^2\frac{d^2u}{dX^2} + (8m-2)X\frac{du}{dX} + [4m^2 - 6m + (1-\alpha)X]u = 0 \tag{13}$$

By introducing the further transform $z = X^\beta$ where β is a real number, (13) assumes the form

$$4\beta^2X^{2\beta}\frac{d^2u}{dz^2} + \left[4\beta(\beta-1)X^\beta + (8m-2)\beta X^\beta\right]\frac{du}{dz} + \left[4m^2 - 6m + (1-\alpha)X\right]u = 0$$

This equation may be simplified by the choice $\beta = \frac{1}{2}$ and $m = \frac{3}{4}$ which results in the differential equation

$$z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} + \left[(1-\alpha)z^2 - \left(\frac{3}{2}\right)^2\right]u = 0 \tag{14}$$

Now if we let $(1-\alpha)^{1/2}z = w$ then (14) becomes

$$w^2\frac{d^2u}{dw^2} + w\frac{du}{dw} + \left[w^2 - \left(\frac{3}{2}\right)^2\right]u = 0 \tag{15}$$

which is the Bessel equation of order $\frac{3}{2}$.

The differential equation (15) has linearly independent solutions $J_{\frac{3}{2}}(w)$ and $J_{-\frac{3}{2}}(w)$ which are Bessel functions. The general solution to (15) can therefore be written as

$$u = aJ_{\frac{3}{2}}(w) + bJ_{-\frac{3}{2}}(w) \tag{16}$$

where a and b are arbitrary constants. It is well known that the Bessel functions of half-integer order can be written in terms of the elementary trigonometric functions. In our case we obtain the following explicit forms for $J_{\frac{3}{2}}$ and $J_{-\frac{3}{2}}$ in terms of elementary trigonometric sine and cosine functions:

$$\begin{aligned} J_{\frac{3}{2}}(\sqrt{1-\alpha}z) &= \sqrt{\frac{2}{\sqrt{1-\alpha}\pi z}}\left[\frac{\sin\sqrt{1-\alpha}z}{\sqrt{1-\alpha}z} - \cos\sqrt{1-\alpha}z\right] \\ J_{-\frac{3}{2}}(\sqrt{1-\alpha}z) &= -\sqrt{\frac{2}{\sqrt{1-\alpha}\pi z}}\left[\frac{\cos\sqrt{1-\alpha}z}{\sqrt{1-\alpha}z} + \sin\sqrt{1-\alpha}z\right] \end{aligned}$$

Then the general solution to the field equation (5c) may be written as

$$
\begin{aligned}
y(x) &= (1-\alpha)^{-3/4}\left[\left(c_1+c_2\sqrt{1+x}\right)\sin\sqrt{(1-\alpha)(1+x)}\right. \\
&\quad \left. +\left(c_2-c_1\sqrt{1+x}\right)\cos\sqrt{(1-\alpha)(1+x)}\right]
\end{aligned}
$$

where we have introduced the new constants $c_1 = a\sqrt{\frac{2}{\pi}}$ and $c_2 = -b\sqrt{\frac{2}{\pi}}$ for simplicity.

As $y(x)$ is now determined the forms for ρ and p may then be established via (5a) and (5b). The complete solution of the Einstein–Maxwell system (5) for this configuration, in terms of the radial coordinate r, is thus given by the system

$$
\begin{aligned}
e^{\lambda} &= \sqrt{1+Cr^2} && \text{(17a)} \\
e^{\nu} &= \frac{A}{(1-\alpha)^{3/4}}\left[\left(c_1+c_2\sqrt{1+Cr^2}\right)\sin f(r)+\left(c_2-c_1\sqrt{1+Cr^2}\right)\cos f(r)\right] && \text{(17b)} \\
\frac{\rho}{C} &= \frac{(2-\alpha)Cr^2+6}{2(1+Cr^2)^2} && \text{(17c)} \\
\frac{p}{C} &= \left[\left(\beta(2-\alpha)(1+Cr^2)^{3/2}+(\alpha+2+4\tilde{\alpha})(1+Cr^2)+\beta(\alpha+4\tilde{\alpha})\sqrt{1+Cr^2}-\alpha\right)\right. \\
&\quad +\left((\alpha-2)(1+Cr^2)^{3/2}+\beta(\alpha+2+4\tilde{\alpha})(1+Cr^2)\right. \\
&\quad \left.\left. -(\alpha+4\tilde{\alpha})\sqrt{1+Cr^2}-\alpha\beta\right)\tan f(r)\right]\times \\
&\quad \left[-2(1+Cr^2)^2\left((\beta\sqrt{1+Cr^2}-1)-(\beta+\sqrt{1+Cr^2})\tan f(r)\right)\right]^{-1} && \text{(17d)} \\
E^2 &= \frac{\alpha C^2r^2}{(1+Cr^2)^2} && \text{(17e)} \\
\sigma^2 &= \frac{\alpha C^2(3+Cr^2)^2}{(1+Cr^2)^5} && \text{(17f)}
\end{aligned}
$$

where we have set

$$
\beta = \frac{c_1}{c_2}, \;\; \tilde{\alpha} = \sqrt{1-\alpha}-1, \;\; f(r) = \sqrt{(1-\alpha)(1+Cr^2)}.
$$

The line element for this solution has the form

$$
\begin{aligned}
ds^2 = &-\frac{A^2}{(1-\alpha)^{3/2}}\left[\left(c_1+c_2\sqrt{1+Cr^2}\right)\sin\sqrt{(1-\alpha)(1+Cr^2)}+\right. \\
&\left.\left(c_2-c_1\sqrt{1+Cr^2}\right)\cos\sqrt{(1-\alpha)(1+Cr^2)}\right]^2 dt^2+(1+Cr^2)dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2) \quad (18)
\end{aligned}
$$

We believe that this model is a new solution of the coupled Einstein–Maxwell system (5). It is easy to verify that on setting $\alpha = 0$ (i.e. $\tilde{\alpha} = 0$) we regain the Finch and Skea [4] solution for an uncharged sphere. From (17e) we have that $E = 0$ as $\alpha = 0$. The Finch and Skea solution is then given by

$$
\begin{aligned}
e^{\lambda} &= \sqrt{1+Cr^2} && \text{(19a)}\\
e^{\nu} &= A\left[(c_1+c_2\sqrt{1+Cr^2})\sin\sqrt{1+Cr^2}+(\,c_2-c_1\sqrt{1+Cr^2}\,)\,\cos\sqrt{1+Cr^2}\right] && \text{(19b)}\\
\rho &= \frac{3+Cr^2}{(1+Cr^2)^2} && \text{(19c)}\\
p &= -\frac{C}{1+Cr^2}\frac{\left(\beta\sqrt{1+Cr^2}+1\right)+\left(\beta-\sqrt{1+Cr^2}\right)\tan\sqrt{1+Cr^2}}{\left(\beta\sqrt{1+Cr^2}-1\right)-\left(\beta+\sqrt{1+Cr^2}\right)\tan\sqrt{1+Cr^2}} && \text{(19d)}
\end{aligned}
$$

The uncharged solution (19) has been extensively studied by Finch and Skea and shown to be regular in the interior of the star and matches smoothly to the Schwarzschild exterior at the boundary. Furthermore, their solution has been shown to be consistent with the neutron star models proposed in the the theory of Walecka [9]. Our Einstein-Maxwell solution (17) is a charged generalisation of the physically reasonable model (19) and does reduce to it when $\alpha = 0$.

3.3 The case $\alpha > 1$

We now briefly consider the case $\alpha > 1$, so that $1-\alpha$ is negative, in the differential equation (9). It is convenient to introduce the new constant ψ so that

$$1-\alpha = -\psi^2$$

As the integration of (9) when $\alpha > 1$ is similar to that in §3.2 we omit the details. The equivalent of equation (14) is

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}-\left[\psi^2 z^2+\left(\frac{3}{2}\right)^2\right]u = 0 \tag{20}$$

If we introduce the new independent variable $\psi z = \tilde{w}$ then equation (20) becomes

$$\tilde{w}^2\frac{d^2u}{d\tilde{w}^2}+\tilde{w}\frac{du}{d\tilde{w}}-\left[\tilde{w}^2+\left(\frac{3}{2}\right)^2\right]u = 0 \tag{21}$$

We recognize (21) as a special case of the modified Bessel equation of fractional order. It has linearly independent solutions, in terms of the modified Bessel functions $I_{\frac{3}{2}}$ and $I_{-\frac{3}{2}}$, which may be expressed in terms of hyperbolic functions. These are given by

$$
\begin{aligned}
I_{\frac{3}{2}}(\psi z) &= \sqrt{\frac{2}{\psi\pi z}}\left(-\frac{\sinh(\psi z)}{\psi z}+\cosh(\psi z)\right)\\
I_{-\frac{3}{2}}(\psi z) &= \sqrt{\frac{2}{\psi\pi z}}\left(\sinh(\psi z)-\frac{\cosh(\psi z)}{\psi z}\right)
\end{aligned}
$$

The general solution to (21) is thus given by

$$u = aI_{\frac{3}{2}}(\psi z) + bI_{-\frac{3}{2}}(\psi z) \tag{22}$$

where a and b are constants. The equivalent of (22) is (16) for $0 \leq \alpha < 1$. We can now obtain the gravitational potential $y(x)$ in the form

$$y(x) = \left(c_1\sqrt{1+x} - c_2\psi\right)\sinh\left(\psi\sqrt{1+x}\right) + \left(c_2\sqrt{1+x} - c_1\psi\right)\cosh\left(\psi\sqrt{1+x}\right)$$

which is the general solution of (9) for $\alpha > 1$. Note that we have introduced the new constants $c_1 = b\sqrt{\frac{2}{\pi}}$ and $c_2 = a\sqrt{\frac{2}{\pi}}$ in $y(x)$.

The complete solution to the Einstein-Maxwell equations (5) can now be determined. In terms of the radial coordinate r it is given by

$$\begin{aligned}
e^{\lambda} &= \sqrt{1+Cr^2} && \text{(23a)}\\
e^{\nu} &= A\Big[\left(c_1\sqrt{1+Cr^2} - c_2\psi\right)\sinh(\psi\sqrt{1+Cr^2}) \\
&\quad + \left(c_2\sqrt{1+Cr^2} - c_1\psi\right)\cosh(\psi\sqrt{1+Cr^2})\Big] && \text{(23b)}\\
\rho &= \frac{(1-\psi^2)\,C^2r^2 + 6C}{2(1+Cr^2)^2} && \text{(23c)}\\
p &= \frac{(\psi^2-1)C^2r^2 - 2C}{2(1+Cr^2)^2} + \frac{2}{1+Cr^2}\times \\
&\quad \frac{\left[\psi^2\left(\tanh(\psi\sqrt{1+Cr^2}) + \beta\right)\right]}{\left[\left(\beta\psi\sqrt{1+Cr^2} - 1\right)\tanh(\psi\sqrt{1+Cr^2}) + \left(\psi\sqrt{1+Cr^2} - \beta\right)\right]} && \text{(23d)}\\
E^2 &= \frac{(\psi^2+1)C^2r^2}{(1+Cr^2)^2} && \text{(23e)}\\
\sigma^2 &= \frac{(\psi^2+1)C^2(3+Cr^2)^2}{(1+Cr^2)^5} && \text{(23f)}
\end{aligned}$$

where we have put $\beta = c_1/c_2$ and $\alpha = \psi^2 + 1$. The line element for this solution is given by

$$\begin{aligned}
ds^2 &= A\left[\left(c_1\sqrt{1+Cr^2} - c_2\psi\right)\sinh(\psi\sqrt{1+Cr^2}) + \left(c_2\sqrt{1+Cr^2} - c_1\psi\right)\cosh\psi\sqrt{1+Cr^2}\right]dt^2 \\
&\quad + (1+Cr^2)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned} \tag{24}$$

It is interesting to observe that this solution is similar to our charged analogue of the Finch and Skea solution generated in §3.2. However in this case it is not possible to eliminate the electric field E (except at the centre), in order to obtain an uncharged counterpart, since $\psi^2 + 1 > 0$. This means that this solution models a sphere that is always charged and hence cannot attain a neutral state. This feature is also shared by other charged solutions including that of Patel and Mehta [10].

In summary the Einstein-Maxwell models represented by the interior metrics (11), (18) and (24) are three new classes of exact solutions to the field equations which are all expressible in terms of elementary functions.

4 Physical considerations: $0 \leq \alpha < 1$

We now study the physical properties of the solution (17) corresponding to $0 \leq \alpha < 1$ which is the uncharged analogue of the Finch and Skea neutron star model. Following Finch and Skea [4] we introduce the substitution

$$v = \sqrt{1 + Cr^2}$$

in our solution (17) and obtain the following forms for the energy density ρ, the pressure p and the electric field intensity E:

$$\frac{\rho}{C} = \frac{(2-\alpha)v^2 + \alpha + 4}{2v^4} \tag{25a}$$

$$\begin{aligned} \frac{p}{C} = {} & \Big[\big((2-\alpha)\beta v^3 + (\alpha - 2 + 4\sqrt{1-\alpha})v^2 + (\alpha - 4 + 4\sqrt{1-\alpha})\beta v - \alpha\big) + \\ & \big((\alpha-2)v^3 + (\alpha - 2 + 4\sqrt{1-\alpha})\beta v^2 - (\alpha - 4 + 4\sqrt{1-\alpha})v - \alpha\beta\big)\tan\sqrt{1-\alpha}v\Big] \\ & \times\left[-2v^4\left((\beta v - 1) - (\beta + v)\tan\sqrt{1-\alpha}v\right)\right]^{-1} \end{aligned} \tag{25b}$$

$$\frac{E^2}{C} = \frac{\alpha(v^2-1)}{v^4} \tag{25c}$$

in terms of the new variable v.

The gravitational behaviour of this model is difficult to analyse because of the complexity of the expressions for ρ and p. In addition the solution is dependent on two parameters α and β; it is necessary to select an appropriate value for one of the parameters and then determine the second parameter from physical considerations. Specifying a value for α is not a simple matter. Some values of α lead to unphysical behaviour (e.g. the value $\alpha = \frac{3}{4}$ results in the undesirable feature that $p_0 = 0$ for all choices of β). Note that the subscript in p_0 denotes the centre $r = 0$. In the present discussion we select the value

$$\alpha = \frac{17}{81}$$

for the physical analysis of the matter and gravitational variables. This value of α chosen is consistent with the interval of validity for α ($0 \leq \alpha < 1$) and it can be demonstrated that this particular value of α does not restrict the value of the parameter β in general. The matter variables (25) assume the following simpler forms for this choice of α:

$$\frac{\rho}{C} = \frac{145v^2+341}{162v^4} \tag{26a}$$

$$\frac{p}{C} = \frac{(145\beta v^3+143v^2-19\beta v-17)-(145v^3-143\beta v^2-19v+17)\tan\left(\frac{8}{9}v\right)}{162v^4\left[(1-\beta v)+(\beta+v)\tan\left(\frac{8}{9}v\right)\right]} \tag{26b}$$

$$\frac{E^2}{C} = \frac{17(v^2-1)}{81v^4} \tag{26c}$$

It is interesting to note that the Einstein-Maxwell solution (26) satisfies a barotropic equation $p = p(\rho)$ which may be determined explicitly. The pressure p is given in terms of the energy density by

$$\begin{aligned}\frac{p}{C} &= \left(\left[145\tilde{\rho}^{3/2}+143\tilde{\rho}-19\tilde{\rho}^{1/2}-17\right]\right.\\ &- \left.\left[145\tilde{\rho}^{3/2}-143\tilde{\rho}-19\tilde{\rho}^{1/2}+17\right]\tan\left(\frac{8}{9}\tilde{\rho}^{1/2}\right)\right)\\ &\times \left(162\tilde{\rho}^2\left[\left(1-\tilde{\rho}^{1/2}\right)+\left(1+\tilde{\rho}^{1/2}\right)\tan\left(\frac{8}{9}\tilde{\rho}^{1/2}\right)\right]\right)^{-1}\end{aligned} \tag{27}$$

where we have set

$$\tilde{\rho} = \left(\frac{145C+\sqrt{21025C^2+220968C\rho}}{324\rho}\right)^{1/2}$$

for convenience. Thus (27), with $\alpha = \frac{17}{81}$, represents a simple equation of state for a charged star.

With the value of $\alpha = \frac{17}{81}$ the rate of change of the energy density, pressure and the electric field intensity are given by:

$$\frac{d\rho}{Cdv} = \frac{-682-145v^2}{81v^5} \tag{28a}$$

$$\begin{aligned}\frac{dp}{Cdv} &= \left(v\left[1-\beta v+(\beta+v)\tan\left(\frac{8}{9}v\right)\right]\left[435\beta v^2+286v-19\beta\right.\right.\\ &\left.-(435v^2-286\beta v-19)\tan\left(\frac{8}{9}v\right)-\frac{8}{9}(145v^3-143\beta v^2-19v+17\beta)\sec^2\left(\frac{8}{9}\right)v)\right]\\ &-\left[145\beta v^3+143v^2-19\beta v-17-(145v^3-143\beta v^2-19v+17\beta)\tan\left(\frac{8}{9}v\right)\right]\times\\ &\left.\left[4-5\beta v+(4\beta+5v)\tan\left(\frac{8}{9}v\right)+\frac{8}{9}v(\beta+v)\sec^2\left(\frac{8}{9}v\right)\right]\right)\Big/\\ &\left(v^5\left[1-\beta v+(\beta+v)\tan\left(\frac{8}{9}v\right)\right]^2\right)\end{aligned} \tag{28b}$$

$$\frac{1}{\sqrt{C}}\frac{dE}{dv} = \frac{\sqrt{\alpha}(2-v^2)}{v^3\sqrt{v^2-1}} \tag{28c}$$

Using (28) we obtain the following expression for the speed of sound:

$$
\begin{aligned}
\frac{dp}{d\rho} \;=\; & \left(v\left[1-\beta v+(\beta+v)\tan\left(\frac{8}{9}v\right)\right]\left[(435\beta v^2+286v-19\beta)\right.\right. \\
& \left.-(435v^2-286\beta v-19)\tan\left(\frac{8}{9}v\right)-\frac{8}{9}(145v^3-143\beta v^2-19v+17\beta)\sec^2\left(\frac{8}{9}\right)v)\right] \\
& -\left[145\beta v^3+143v^2-19\beta v-17-(145v^3-143\beta v^2-19v+17\beta)\tan\left(\frac{8}{9}v\right)\right]\times \\
& \left.\left[4-5\beta v+(4\beta+5v)\tan\left(\frac{8}{9}v\right)+\frac{8}{9}v(\beta+v)\sec^2\left(\frac{8}{9}v\right)\right]\right)\Big/ \\
& \left((-290v^2-1364)\left[(1-\beta v+(\beta+v)\tan\left(\frac{8}{9}v\right)\right]^2\right) \qquad (29)
\end{aligned}
$$

For a regular model we require that $\frac{d\rho}{dv}$, $\frac{dp}{dv}$ and $\frac{dE}{dv}$ are well behaved in the interior of the charged star.

To ensure that the star is regular at $r=0$ it is important to examine the behaviour of our model at the centre of the star. The quantities (26), (28) and (29) have the following forms at the centre of the star:

$$\frac{\rho_0}{C} = 3 \qquad (30a)$$

$$\frac{p_0}{C} = \frac{2.232\beta-0.104}{\beta+9.620} \qquad (30b)$$

$$\left(\frac{d\rho}{Cdv}\right)_0 = \frac{-827}{81} \qquad (30c)$$

$$\left(\frac{dp}{Cdv}\right)_0 = -\frac{363.79\beta^2+1198.15\beta+772.6}{(0.232\beta+2.232)^2} \qquad (30d)$$

$$\left(\frac{dp}{d\rho}\right)_0 = \frac{0.22\beta^2+0.723\beta+0.905}{(0.232\beta+2.232)^2} \qquad (30e)$$

The quantity E vanishes at $r=0$. In order to guarantee the positivity of the central pressure we obtain the following restriction on β from (30b):

$$\beta \leq -9.62 \quad \text{or} \quad \beta \geq 0.104$$

The adiabatic sound speed condition $0 \leq \frac{dp}{d\rho} \leq 1$ yields the additional constraint:

$$-4.06 \leq \beta \leq 5.9$$

for the speed of sound not to exceed the speed of light. The intersection of the above inequalities is given by

$$0.104 \leq \beta \leq 5.9$$

for a regular centre. It is clear that for this range of β the requirements $\rho_0 > 0$, $\left(\frac{d\rho}{dv}\right)_0 < 0$ and $\left(\frac{dp}{dv}\right)_0 < 0$ are trivially satisfied.

Now we are in a position to investigate the gravitational behaviour of our model in the interior of the star. The behaviour of the model is illustrated best in terms of graphs of the matter variables and the gravitational potentials. These graphs have been generated with the assistance of the software package Mathematica. Based on the above limits obtained on β, we select the value

$$\beta = 1.$$

The graphs of the various quantities are on the interval $1 \leq v \leq 1.6$. Figures 1-3 represent the behaviour of the energy density ρ, the pressure p and the electric field intensity E, respectively. In Figure 4 we have plotted $\frac{dp}{d\rho}$ on the interval $1 \leq v \leq 1.6$. The metric functions $e^{2\nu}$ and $e^{2\lambda}$ are plotted on the same interval in Figures 5 and 6, respectively.

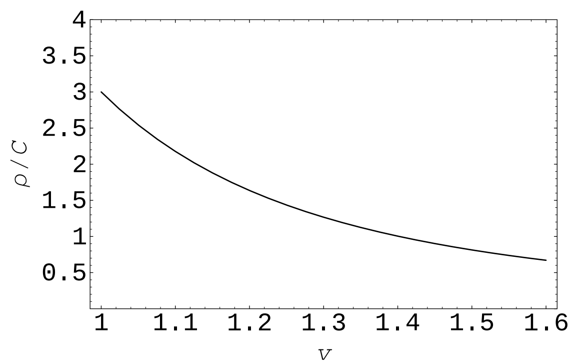

Figure 1: A plot of the energy density ρ/C versus $v = \sqrt{1 + Cr^2}$.

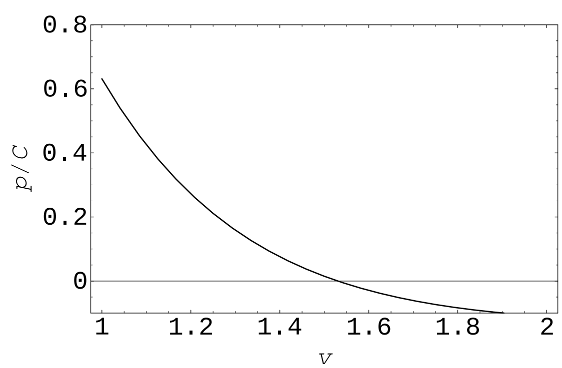

Figure 2: A plot of the pressure p/C versus $v = \sqrt{1 + Cr^2}$.

From the plots in Figures 1-2, we observe that the energy density ρ and the pressure p are positive and monotonically decreasing functions in the interior of the star. We observe from Figure 3 that the electric field is positive and monotonically increasing on $1 \leq v < \sqrt{2}$, has a maximum value at $v = \sqrt{2}$, and then decreases very slowly to the boundary. The potentials

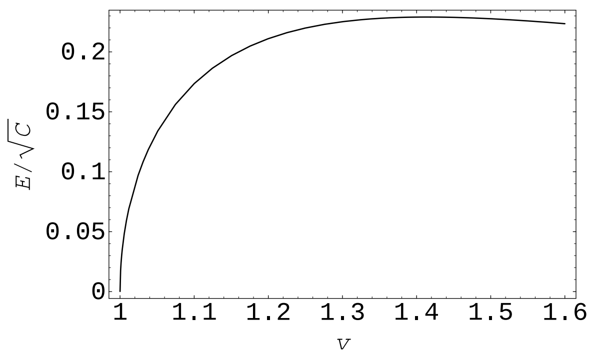

Figure 3: A plot of the electric field intensity $E/\sqrt{C}$ versus $v = \sqrt{1 + Cr^2}$.

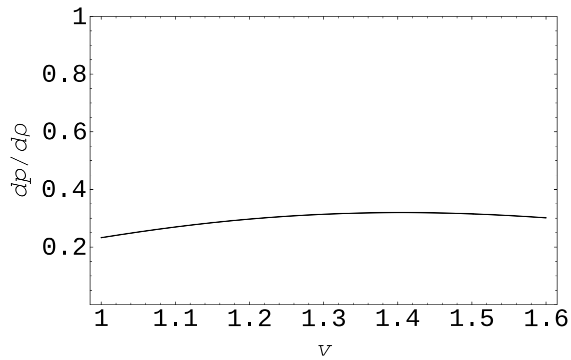

Figure 4: A plot of the speed of sound $dp/d\rho$ versus $v = \sqrt{1 + Cr^2}$.

$e^{2\nu}$ and $e^{2\lambda}$ are regular in the interior of the star as illustrated in Figures 5 and 6 respectively. Thus the quantities ρ, p, E, $e^{2\nu}$ and $e^{2\lambda}$ are continuous, regular and well behaved throughout the interval $1 \leq v \leq 1.6$. From Figure 2 we note that the pressure vanishes at approximately

$$v = 1.5252 \tag{31}$$

which fixes the boundary of the charged star. The value (31) is consistent with the value of v for which $p(v) = 0$ where p is given by (30b). Of course the boundary of the star will change for other values of α and β. A pleasing feature of this model is the behaviour of the speed of sound. It can be observed from Figure 4 that the speed of sound is always less than unity everywhere for $1 \leq v \leq 1.5252$. Therefore the causality principle $0 \leq dp/d\rho \leq 1$ holds throughout the star and the speed of sound is always less than the speed of light. Thus our solution (17), with $\alpha = \frac{17}{81}$ and $\beta = 1$, satisfies the requirements for a physically reasonable charged star.

Utilising the information obtained from the graphs we are now able to examine the boundary conditions. In particular we have to match the interior (18) to the Reissner-Nordstrom exterior (6). With the help of $v = 1.5252 = \sqrt{1 + CR^2}$, the vanishing surface pressure condition $p(R) = 0$, yields the following approximate dependence of the constant C on the radius R of

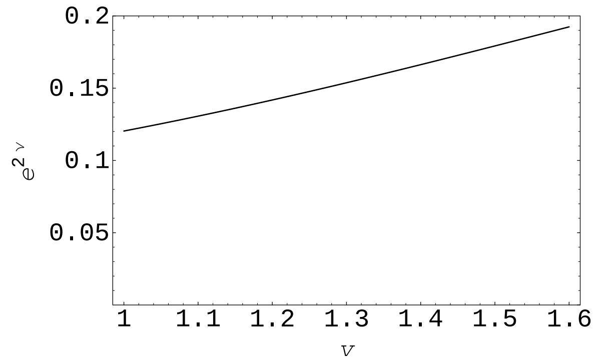

Figure 5: A plot of the gravitational potential $e^{2\nu}$ versus $v = \sqrt{1+Cr^2}$.

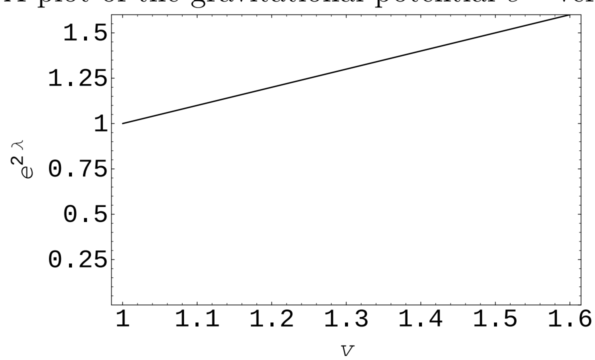

Figure 6: A plot of the gravitational potential $e^{2\lambda}$ versus $v = \sqrt{1+Cr^2}$.

the star:

$$C = \frac{1.33}{R^2} \tag{32}$$

The continuity of the electric field across the boundary condition $E(R) = Q/R^2$ gives the relationship of the total charge Q of the star, as measured by an observer at infinity, with the radius R for $\alpha = 17/81$:

$$Q = \frac{\sqrt{17}R^3}{9(1+R^2)}$$

Then we are in a position to find the charge to radius ratio

$$\frac{Q}{R} = 0.26 \tag{33}$$

On matching of the interior gravitational potentials $e^{2\nu}$ and $e^{2\lambda}$ from (18) with the exterior Reissner–Nordstrom solution (6) we obtain

$$\frac{1}{1+CR^2} = 1 - \frac{2M}{R} + \frac{Q^2}{R^2} \tag{34a}$$

$$\frac{729}{512}c_1\left[(1+\sqrt{1+CR^2})\sin\left(\frac{8}{9}\sqrt{1+CR^2}\right) + (1-\sqrt{1+CR^2})\cos\left(\frac{8}{9}\sqrt{1+CR^2}\right)\right]^2 = \frac{1}{1+CR^2} \tag{34b}$$

On substituting (32) in (34b) we find the following values for the constants c_1 and c_2:

$$c_1 = 0.05444 = c_2$$

while (34a) gives another expression for C:

$$C = \frac{2MR - Q^2}{R^2(R^2 - 2MR + Q^2)} \tag{35}$$

in terms of Q. Note that on setting $Q = 0$ we obtain the same value of C as in the uncharged model of Finch and Skea [4]. Therefore the introduction of the electromagnetic field has the effect of decreasing the value of C; consequently the radius R of the charged star is affected via the relationship (32). Substituting for C from (32) and for Q from (33) in (35) we obtain the following mass-radius relationship:

$$\frac{M}{R} = 0.3 \tag{36}$$

Note that the Buchdahl limit

$$M/R < 4/9$$

for stability is applicable for neutral stars. We observe that this limit is not violated in our solution (17) in the presence of charge. In addition from (33) and (36) we obtain the following mass-charge ratio

$$\frac{M}{Q} = 1.154$$

which satisfies the condition for equilibrium

$$M^2 > Q^2$$

of charged stars with nonzero pressure in general as pointed out by Cooperstock and de la Cruz [11].

5 Conclusion

In this paper we obtained three categories of solutions for the Einstein-Maxwell system (5) corresponding to the electric field intensity (8). In the first category of solutions $E \neq 0$ (except at the centre) and the solutions can be written in terms of elementary functions. In the second category of solutions it is possible for $E = 0$ throughout the interior spacetime and we consequently regain the Finch and Skea solution [4] which is a good model for a neutron star. The charged analogue of the Finch-Skea solution is given in terms of Bessel functions of half-integer order. We demonstrated that this charged solution satisfies an explicit barotropic equation of state. In the third category of solutions $E \neq 0$ (except at the centre) and the solutions can be written in terms of modified Bessel functions of half-integer order. A detailed analysis of the charged analogue of the Finch-Skea solution was carried out. For specific parameter values the energy density ρ, the pressure p, the electric field intensity E, the speed of sound $dp/d\rho$, the gravitational potential $e^{2\nu}$ and the metric function $e^{2\lambda}$ were plotted. The profiles of the matter, electromagnetic and gravitation variables suggest that these quantities are well behaved so that the charged analogue of the Finch-Skea solution describes a realistic charged stellar body. It was shown that the speed of sound is less than the speed of light and consequently causality is not violated. In particular we observe that the mass-radius ratio is within the Buchdahl limit for uncharged stars and the mass-charge ratio is consistent with the requirements of Cooperstock and de la Cruz [11]. As this model contains the physically acceptable Finch and Skea solution and appears to be physically viable, a detailed analysis of the stability should be carried out; this is the subject of ongoing research.

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